Ponderings and musings on discrete exterior calculus

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I wish to celebrate Cesare's birthday by sharing with him and his guests some ruminations of mine on matters loosely related to part of his own scientific interests and achievements.

A cursory Wikipedia article defines my topic of choice—discrete exterior calculus—as "the extension of the exterior calculus to discrete spaces including graphs and finite element meshes." Shouldn't that be the other way round? Shouldn't calculus on manifolds emerge as an asymptotic extension of combinatorial operations performed in discrete spaces?

In a discrete setting, chains and their boundaries are primary. Cochains and coboundaries come next, by duality. Differential forms and exterior derivatives emerge from these latter in the limit of infinite refinement. Why then are they primary in differential geometry? What are the analogs of chains and boundaries in a differentiable manifold? Why they seem to be missing?

It is common sense that the algebraic properties of multivectors and exterior forms are preliminary to—and more straightforward than—the machinery of exterior calculus. However, in discrete spaces just the opposite appears to be true. Boundary and coboundary operators come out cleanly and effortlessly—with a discrete Stokes theorem for free as a bonus. Contrariwise, introducing basic algebraic constructions such as the wedge product or the Hodge star in a discrete setting is awkward and fraught with difficulties. Why is it like that?